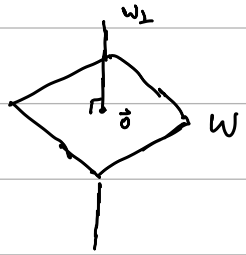
# Lecture 25

$\boxed{\text{Ex}}$ $W = \text{span}\left\{ \begin{pmatrix}1\\0\\2\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix} \right\} \subset \mathbb{R}^3$

What is a basis for $W^\perp$?




Soln: $x \in \vec{W}^\perp$ i.f.f. $\vec{x} \cdot \begin{pmatrix}1\\0\\2\end{pmatrix} = \vec{x} \cdot \begin{pmatrix}2\\1\\1\end{pmatrix} = 0$

i.e. $\vec{x}$ solves $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right]$ where $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \end{bmatrix}$

So $W^\perp = \ker A$

$\to \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right] \longrightarrow \vec{x} = x_3 \begin{pmatrix}-2\\3\\1\end{pmatrix}$

$\cdot$ So $\begin{pmatrix}-2\\3\\1\end{pmatrix}$ is a basis for $W^\perp$

i.e. $W^\perp = \text{span}\begin{pmatrix}-2\\3\\1\end{pmatrix}$

<u>Moral</u>: $A$ = matrix $\longrightarrow$ $\ker A$ = orthog complement of "span of rows of $A$"

= "row space of $A$"

= Span of cols of $A^T$

= "col space of $A^T$"

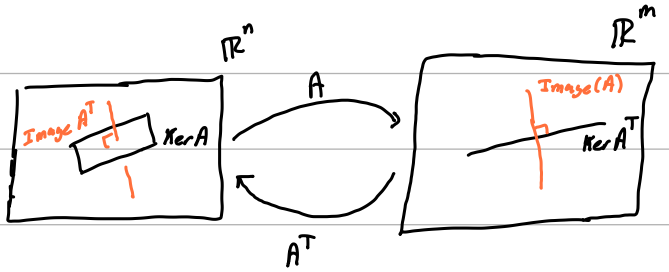
= image $(A)$

Prop : $\forall$ matrix $A$

$$\ker(A) = (\text{Image } A^T)^{\perp}$$

and likewise $\ker(A^T) = (\text{Image } A)^{\perp}$

$A$ = $m \times n$ matrix




- $A$ takes every vec of image$(A^T)$ to image$(A)$
- $A^{-1}$ takes every vec of image$(A)$ to image$(A^T)$

- Cor : $\text{rank}(A) = \text{rank}(A^T)$

Proof : $\text{rank}(A) := \dim[\text{image}(A)]$

by rank theorem : # of columns of $A$ $- [\dim(\ker A)]$

on the other hand : $\text{rank } A^T = \dim[\text{image}(A^T)] = \dim(\ker A)^{\perp} = n - \dim(\ker A)$

$\therefore$ rank $A$ = rank $A^T$ ∎

- Altnerative way to say corelary :
  - dim of col space of $A$ is the dimension of row space of $A$.

# 5.4 Least Squares solutions of linear systems

$\boxed{\text{Ex}}$ What point in $\mathbb{R}^2$ comes closest to lying on

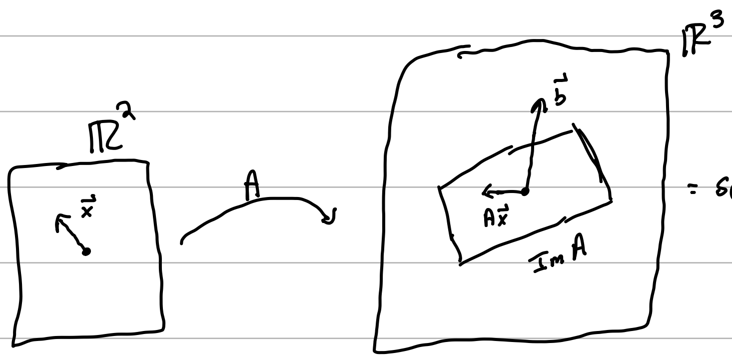
$$x+y=4$$
$$2x-y=0$$
$$3x-y=1$$

Solution:

- Want to solve:

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 3 & -1 \end{bmatrix}} \; \underset{\vec{b}}{\begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}} \to \cdots \to \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 1 & 8/3 \\ 0 & 0 & -1/3 \end{array}\right]$$

Third row says this is inconsistent




$= \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \right\}$

Def: We call $x^* \in \mathbb{R}^n$ a <u>least squares solution</u> of $A\vec{x}=\vec{b}$ if the length $\|A\vec{x}^* - \vec{b}\|$ is as small as possible.

Idea: Choose $\vec{x}^*$ so $A\vec{x}^* = \text{proj}_{Im(A)} \vec{b}$

Theorem: Let $W \subset \mathbb{R}^n$ be a subspace and $\vec{v} \in \mathbb{R}^n$ a vector. Then the closest vector in $W$ to $\vec{v}$ is $proj_W(\vec{v})$.

I.e. $\forall\, \vec{w} \in W$, we have $\|\vec{w} - \vec{v}\| \geq \|proj_W(\vec{v}) - \vec{v}\|$